The Equations of Almost Complete Intersections

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Abstract

In this paper we examine the role of four Hilbert functions in the determination of the defining relations of the Rees algebra of almost complete intersections of finite colength. Because three of the corresponding modules are Artinian, some of these relationships are very effective, opening up tracks to the determination of the equations and also to processes of going from homologically defined sets of equations to higher degrees ones assembled by resultants.

1 Introduction

Let R be a Noetherian ring and let I be an ideal. By the *equations* of I it is meant a free presentation of the Rees algebra $R[It]$ of I,

$$0 \to L \longrightarrow S = R[\mathbf{T}_1, \ldots, \mathbf{T}_m] \xrightarrow{\psi} R[It] \to 0, \quad \mathbf{T}_i \mapsto f_i t. \tag{1}$$

More precisely, L is the defining ideal of the Rees algebra of I but we refer to it simply as the *ideal of equations* of I. The ideal L depends on the chosen set of generators of I, but all of its significant

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cohomological properties, such as the integers that bound the degrees of minimal generating sets of L, are independent of the presentation ψ. The examination of L is one pathway to the unveiling of the properties of $R[It]$. It codes the syzygies of all powers of I, and therefore is a carrier of not just algebraic properties of I, but of analytic ones as well. It is also a vehicle to understanding geometric properties of several constructions built out of I, particularly of rational maps.

The search for these equations and their use has attracted considerable interest by a diverse group of researchers. We just mention some that have directly influenced this work. One main source lies in the work of L. Busé, M. Chardin, D. Cox and J. P. Jouanolou who have charted, by themselves or with co-workers, numerous roles of resultants and other elimination techniques to obtain these equations ([1], [2], [3], [4], and references therein.) Another important development was given by A. Kustin, C. Polini and B. Ulrich, who provided a comprehensive analysis of the equations of ideals (in the binary case), not just necessarily of almost complete intersections, but still of ideals whose syzygies are almost all linear ([11]). Last, has been the important work of D. Cox not only for its theoretical value to the understanding of these equations, but for the role it has played in bridging the fields of commutative algebra and of geometric modelling ([5], [6]).

This is a sequel, although not entirely a continuation of [9]. It deals, using some novel methods, with questions in higher dimensions that were triggered in that project, but is mainly concerned with the more general issues of the structure of the Rees algebras of almost complete intersections. Our underlying metaphor here is to focus on distinguished sets of equations by examining 4 Hilbert functions associated to the ideal I and to the coefficients of ts syzygies. It brings considerable effectivity to the methods by developing explicit formulas for some of the equations.

There are natural and technical reasons to focus on almost intersections. A good deal of elimination theory is intertwined with birationality questions. Now, a regular sequence of forms of fixed degree ≥ 2 never defines a birational map. Thus, the first relevant case is the next one, namely, that of an almost complete intersection. Say, $I \subset R = k[x_1, \ldots, x_d]$ is minimally generated by forms $a_1, \ldots, a_d, a_{d+1}$ of fixed degree, where $a_1, \ldots, a_d$ form a regular sequence. If these generators define a birational map of $\mathbb{P}^{d-1}$ onto its image in $\mathbb{P}^d$ then any set of forms of this degree containing $a_1, \ldots, a_d, a_{d+1}$ still defines a birational map onto its image. Thus, almost complete intersections give us in some sense the hard case.

Note that these almost complete intersections have maximal codimension, i.e., the ideal I as above is $\mathfrak{m}$-primary, where $\mathfrak{m} = (x_1, \ldots, x_d)$. The corresponding rational map with such base ideal is a regular map with image a hypersurface of $\mathbb{P}^d$. However, one can stretch the theory to one more case, namely, that of an almost complete intersection of $I = (a_1, \ldots, a_{d-1}, a_d)$ of submaximal height $d-1$. Here the corresponding rational map $\Psi_I : \mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^{d-1}$ is only defined off the support $V(I)$ (whose geometric dimension is 0) and one can ask when this map is birational—thus corresponding to the notion of a Cremona transformation of $\mathbb{P}^{d-1}$. In another paper, we will treat these ideals.

Notation 1.1 To describe the problems treated and the solutions given, we give a modicum of notation and terminology. When not obvious, we will point out which characteristics of k to avoid.

- $R = k[x_1, \ldots, x_d]_{\mathfrak{m}}$ with $d \geq 2$ and $\mathfrak{m} = (x_1, \ldots, x_d)$.
- $I = (a_1, \ldots, a_d, a_{d+1})$ with $\deg(a_i) = n$ for all i, height$(I) = d$ and I minimally generated by these forms.

- Assume that $J = (a_1, \ldots, a_d)$ is a minimal reduction of $I = (J, a)$, that is $I^{r+1} = JI^r$ for some natural number r.

- $\bigoplus_j R(-n_j) \xrightarrow{\varphi} R^{d+1}(-n) \longrightarrow I \longrightarrow 0$ is a free minimal presentation of I.

- $S = R[\mathbf{T}_1, \ldots, \mathbf{T}_{d+1}]$ and $L = \ker(S \twoheadrightarrow R[It])$ via $T_j \mapsto a_j t$, where $R[It]$ is the Rees algebra of I; note that L is a homogeneous ideal in the standard grading of S with $S_0 = R$.

- L_i : R–module generated by forms of L of degree i in $\mathbf{T}_j$'s. For example, the degree 1 component of L is the ideal of entries of the matrix product

$$(L_1) = I_1([\mathbf{T}_1 \quad \cdots \quad \mathbf{T}_{d+1}] \cdot \varphi).$$

- $\nu(L_i)$ denotes the minimal number of *fresh* generators of L_i. Thus $\nu(L_2)$ is the minimal number of generators of the R-module L_2/S_1L_1.

- The *elimination degree* of I is

$$\operatorname{edeg}(I) \quad = \quad \inf\{i \mid L_i \not\subseteq \mathfrak{m}S\}. \tag{2}$$

  One knows quite generally that $L = (L_1) : \mathfrak{m}^\infty$. A *secondary elimination degree* is an integer r such that $L = (L_1) : \mathfrak{m}^r$, i.e., an integer at least as large as the stabilizing exponent of the saturation.

- The *special fiber* of I is the ring $\mathcal{F}(I) = R[It] \otimes R/\mathfrak{m}$. This is a hypersurface ring

$$\mathcal{F}(I) = k[T_1, \ldots, T_{d+1}]/(\mathbf{f}(\mathbf{T})), \tag{3}$$

  where $\mathbf{f}(\mathbf{T})$ is an irreducible polynomial of degree $\operatorname{edeg}(I)$. Then $\mathbf{f}(\mathbf{T})$ is called the *elimination equation* of I, and may be taken as an element of L.

Besides the syzygies of I, $\mathbf{f}(\mathbf{T})$ may be considered the most significant of the *equations* of I. Determining it, or at least its degree, is one of the main goals of this paper. The enablers, in our treatment, are four Hilbert functions associated to I, those of R/I, $R/J : a$, $R/I_1(\varphi)$ and the Hilbert-Samuel function defined by I. Each encodes, singly or in conjunction, a different aspect of L.

In order to describe the other relationships between the invariants of the ideal I and its equations, we make use the following diagram:

$$
\begin{tikzcd}
 & & I \arrow[r] \arrow[dl] \arrow[dr] & e_1(I) \arrow[r] & \Psi_I \text{ birational?} \\
 & J:a \arrow[dl] \arrow[d] & & I_1(\varphi) \arrow[d] & \\
L=(L_1):\mathfrak{m}^r & L_1 \arrow[l] \arrow[dr] & & L_2 \arrow[dl] & \\
 & & \text{Res }(L_1,L_2) \arrow[r] & \text{elim. eq.} &
\end{tikzcd}
$$

At the outset and throughout there is the role played by the Hilbert function of $J:a$, which besides that of directing all the syzygies of I, is the encoding of an integer r such that $L=(L_1):\mathfrak{m}^r$. According to Theorem 2.6, r can be taken as $r=\epsilon+1$, where ϵ is the *socle degree* of $R/J:a$. Since L is expressed as the quotient of two Cohen-Macaulay ideals, this formula has shown in practice to be an effective tool to determine saturation.

A persistent question is that of how to obtain higher degree generators from the syzygies of I. We will provide an iterative approach to generate the successive components of L:

$$L_1 \rightsquigarrow L_2 \rightsquigarrow L_3 \rightsquigarrow \cdots .$$

This is not an effective process, except for the step $L_1 \rightsquigarrow L_2$. The more interesting development seems to be the use of the syzygetic lemmas to obtain $\delta(I)=L_2/S_1L_1$ in the case of almost complete intersections. This is a formulation of the *method of moving quadrics* of several authors. The novelty here is the use of the Hilbert function of the ideal $I_1(\varphi)$ to understand and conveniently express $\delta(I)$. Such level of detail was not present even in earlier treatments of $\delta(I)$. The *syzygetic lemmas* are observations based on the Hilbert functions of I and of $I_1(\varphi)$ to allow a description of L_2. It converts the expression (8)

$$\delta(I)=\mathrm{Hom}_{R/I}(R/I_1(\varphi),H_1(I)), \tag{4}$$

where $H_1(I)$ is the canonical module of R/I (given by the syzygies of I), into a set of generators of L_2/S_1L_1. It is fairly effective in the case of binary forms, many cases of ternary and some quaternary forms, as we shall see. In these cases, out of L_1 and L_2 we will be able to write the elimination equation in the form of a resultant

$$\text{Res }(L_1,L_2), \tag{5}$$

or as one of its factors. We prove the non-vanishing of this determinant under three different situations. In the case of binary ideals, whose syzygies are of arbitrary degrees, we give a far-reaching generalization of [9]. Here we make use of one of the distinguished submodules of $\delta(I)$,

$$\delta_s(I)=\mathrm{Hom}_R(R/\mathfrak{m}^s,\delta(I))\hookrightarrow \delta(I),$$

where s is the order of the ideal $I_1(\varphi)$. While $\delta(I)$ accounts for the whole of L_2, $\delta_s(I)$ collects the forms to be assembled in a resultant. For an ideal I of $k[x, y]$, generated by 3 forms of degree n, we build out of L_1 and L_2, in a straightforward manner, a nonzero polynomial of $k[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3]$ in L, of degree n (Theorem 3.1).

The other results, in higher dimension, require that the content of syzygies ideal $I_1(\varphi)$ be a power of the maximal ideal $\mathfrak{m}$, that is $\delta_s(I) = \delta(I)$. Thus, in the case of ternary forms of degree n whose content ideal $I_1(\varphi) = \mathfrak{m}^{n-1}$, our main result (Theorem 4.1) proves the non-vanishing of Res (L_1, L_2) without appealing to conditions of genericity (but with degree constraints). A final result is very special to quaternary forms (Theorem 5.1).

2 Syzygies and Hilbert Functions

Before engaging in the above questions proper, we will outline the basic homological and arithmetical data involved in these ideals.

The resolution

The general format of the resolution of I goes as follows. First, note that $J : a$ is a Gorenstein ideal, and that $I = J : (J : a)$. Since Gorenstein ideals, at least in low dimensions, have an amenable structure, it may be desirable to look at $J : a$ as a building block to I and its equations. In this arrangement the syzygies of I will be organized in terms of those of J and of $J : a$.

Let us recall how the resolution of I arises as a mapping cone of the Koszul complex $\mathbb{K}(J)$ and a minimal resolution of $J : a$ (this was first given in [12]; see also [16, Theorem A.139]).

Theorem 2.1 *Let R be a Gorenstein local ring, let $\mathfrak{a}$ be a perfect ideal of height g and let $\mathbb{F}$ be a minimal free resolution of $R/\mathfrak{a}$. Let $\mathbf{z} = z_1, \ldots, z_g \subset \mathfrak{a}$ be a regular sequence, let $\mathbb{K} = \mathbb{K}(\mathbf{z}; R)$ be the corresponding Koszul complex, and let $u\colon \mathbb{K} \to \mathbb{F}$ be the comparison mapping induced by the inclusion $(\mathbf{z}) \subset I$. Then the dual $\mathbb{C}(u^*)[-g]$ of the mapping cone of u, modulo the subcomplex $u_0\colon R \to R$, is a free resolution of length g of $R/(\mathbf{z})\colon \mathfrak{a}$. Moreover, the canonical module of $R/(\mathbf{z})\colon \mathfrak{a}$ (modulo shift in the graded case) is $\mathfrak{a}/(\mathbf{z})$.*

Remark 2.2 For later reference, we point out three observations when the ideal is the above almost complete intersection I.

1. If $J : a = (b_1, \ldots, b_d)$, writing
$$[a_1, \ldots, a_d] = [b_1, \ldots, b_d] \cdot \phi,$$
gives
$$I = (J, \det(\phi)),$$
so that I is a Northcott ideal.

2. If I is a generated by forms of degree n, a choice for J is simply a set of forms $a_1, \ldots, a_d$ of degree n generating a regular sequence. Many of the features of I–such as the ideal $I_1(\varphi)$–can be read off $J : a$. Namely, $I_1(\varphi)$ is the sum of J and the coefficients arising in the expressions of $a(J : a) \subset J$.

3. Since $J : a$ is a Gorenstein ideal, its rich structure in dimension ≤ 3 is fundamental to the study in these cases.

Hilbert functions

There are four Hilbert functions related to the ideal I that are significant in this paper, the first three of the Artinian modules R/I, $R/I_1(\varphi)$ and $R/J : a$, and are therefore of easy manipulation. Their interactions will be a mainstay of the paper.

The first elementary observation, whose proof is well-known as to be omitted, will be useful when we need the Hilbert function of the canonical module of R/I.

Proposition 2.3 *If I is generated by forms of degree n of $k[x_1, \ldots, x_d]$, the Hilbert function of R/I satisfies*

$$\begin{aligned} H_{R/I}(t) &= H_{R/J}(t) - H_{I/J}(t) \\ &= H_{R/J}(t) - H_{R/J:a}(t-n). \end{aligned}$$

The fourth Hilbert function is that of the associated graded ring

$$\mathrm{gr}_I(R) = \bigoplus_{m \geq 0} I^m / I^{m+1}.$$

It affords the Hilbert–Samuel polynomial ($m \gg 0$)

$$\lambda(R/I^{m+1}) = e_0(I)\binom{d+m}{d} - e_1(I)\binom{d+m-1}{d-1} + \text{lower degree terms of } m,$$

where $e_0(I)$ is the multiplicity of the ideal I. A related Hilbert polynomial is that associated to the integral closure filtration $\{\overline{I^m}\}$:

$$\lambda(R/\overline{I^{m+1}}) = \overline{e}_0(I)\binom{d+m}{d} - \overline{e}_1(I)\binom{d+m-1}{d-1} + \text{lower degree terms of } m.$$

For an $\mathfrak{m}$-primary ideal I generated by forms of degree n, $\overline{I^m} = \mathfrak{m}^{nm}$, so the latter coefficients are really invariants of the ideal $\mathfrak{m}^n$ and one has

$$\begin{aligned} e_0(I) &= \overline{e}_0(I) = n^d \\ e_1(I) &\leq \overline{e}_1(I) = \frac{d-1}{2}(n^d - n^{d-1}). \end{aligned}$$

The case of equality $e_1(I) = \overline{e}_1(I)$ has a straightforward (and general) interpretation in terms of the corresponding Rees algebras.

Proposition 2.4 *Let $(R, \mathfrak{m})$ be an analytically unramified normal local domain of dimension d, and let I be an $\mathfrak{m}$-primary ideal. Then $e_1(I) = \overline{e}_1(I)$ if and only if $\mathcal{R}(I)$ satisfies the condition (R_1) of Serre.*

Proof. Let $\mathbf{A} = \mathcal{R}(I)$, set $\mathbf{B}$ for its integral closure. Then $\mathbf{B}$ is a finitely generated $\mathbf{A}$-module. Consider the exact sequence

$$0 \to \mathbf{A} \longrightarrow \mathbf{B} \longrightarrow C = \mathbf{B}/\mathbf{A} \to 0.$$

Since $e_0(I) = \overline{e}_0(I)$, C is a graded $\mathbf{A}$-module of dimension $\leq d$. If $\dim C = d$, its multiplicity $e_0(C)$ is derived from the Hilbert polynomials above as $e_0(C) = \overline{e}_1(I) - e_1(I)$. This sets up the assertion since $\mathbf{A}$ and $\mathbf{B}$ are equal in codimension one if and only if $\dim C < d$. $\square$

This permits stating [9, Proposition 3.3] as follows (see also [7] for degrees formulas).

Proposition 2.5 *Let $R = k[x_1, \ldots, x_d]$ and let $I = (f_1, \ldots, f_{d+1})$ be an ideal of finite colength, generated by forms of degree n. Denote by $\mathcal{F}$ and $\mathcal{F}'$ the special fibers of $\mathcal{R}(I)$ and $\mathcal{R}(\mathfrak{m}^n)$, respectively. The following conditions are equivalent:*

(i) $[\mathcal{F}' : \mathcal{F}] = 1$, *that is, the rational mapping*

$$\Psi_I = [f_1 : f_2 : \cdots : f_{d+1}] : \mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^d$$

is birational onto its image;

(ii) $\deg \mathcal{F} = n^{d-1}$;

(iii) $e_1(I) = \frac{d-1}{2}(n^d - n^{d-1})$;

(iv) $\mathcal{R}(I)$ *is non-singular in codimension one.*

A great deal of this investigation is to determine $\deg \mathcal{F}$, which as we referred to earlier is the *elimination degree* of I (in notation, $\mathrm{edeg}(I)$). Very often this turns into explicit formulas for the elimination equation.

Secondary elimination degrees

A solution to some of questions raised above resides in the understanding of the primary decomposition of (L_1), the defining ideal of the symmetric algebra $\mathrm{Sym}(I)$ of I over $S = R[\mathbf{T}]$. As pointed out earlier, we know that $L = (L_1) : \mathfrak{m}^\infty$ as quite generally a power of I annihilates $L/(L_1)$. The L-primary component is therefore L itself and (L_1) has only two primary components, the other being $\mathfrak{m}S$-primary. On the other hand, according to [9, Proposition 2.2], (L_1) is Cohen-Macaulay so its primary components are of the same dimension. Write

$$(L_1) = L \cap Q,$$

where Q is $\mathfrak{m}S$-primary. It allows for the expression of L as a saturation of (L_1) in many ways. For example, for nonzero $\alpha \in I$ or $\alpha \in I_1(\varphi)$, we have $L = (L_1) : \alpha^\infty$.

We give now an explicit saturation by exhibiting integers r such that $L = (L_1) : \mathfrak{m}^r$, which as we referred to earlier are *secondary elimination degrees*. Finding its least value is one of our goals in individual cases. In the actual practice we have found the computation effective, perhaps because L is given as the quotient of two Cohen-Macaulay ideals, the second generated by monomials.

Theorem 2.6 *Let $R = k[x_1, \ldots, x_d]$ and let $I = (f_1, \ldots, f_{d+1})$ be an ideal of finite colength, generated by forms of degree n. Suppose $J = (f_1, \ldots, f_d)$ is a minimal reduction, and set $a = f_{d+1}$. Let ϵ be the socle degree of $R/(J : a)$, that is the largest integer m such that $(R/(J : a))_m \neq 0$. If $r = \epsilon + 1$, then*

$$L = (L_1) : \mathfrak{m}^r.$$

More precisely, if some form $\mathbf{f}$ of L_i has coefficients in $\mathfrak{m}^r$, then $\mathbf{f} \in (L_1)$.

Proof. The assumption on r means that $(J : a)_m = (\mathfrak{m}^m)_m$ for $m \geq r$, that is if $J : a$ has initial degree d' then

$$\mathfrak{m}^r = \sum_{i \geq d'} (J : a)_i \mathfrak{m}^{r-i}.$$

Now any element $\mathbf{p} \in L$ can be written as

$$\mathbf{p} = \mathbf{h}_p \mathbf{T}_{d+1}^p + \mathbf{h}_{p-1} \mathbf{T}_{d+1}^{p-1} + \cdots + \mathbf{h}_0,$$

where the $\mathbf{h}_i$ are polynomials in $\mathbf{T}_1, \ldots, \mathbf{T}_d$. If $\mathbf{h}_1, \ldots, \mathbf{h}_p$ all happen to vanish, then $\mathbf{h}_0 \in (L_1)$ since $f_1, \ldots, f_d$ is a regular sequence. The proof will consist in reducing to this situation.

To wit, let u be a form of degree r in $\mathfrak{m}^r$. To prove that $u\mathbf{p} \in (L_1)$, we may assume that $u = v\alpha$, with $v \in \mathfrak{m}^{r-i}$ and α a minimal generator in $(J : a)_i$.

We are going to replace $v\alpha\mathbf{p}$ by an equivalent element of L, but of lower degree in $\mathbf{T}_{d+1}$. Since $\alpha \in (J : a)$, there is an induced form $\mathbf{g} \in L_1$

$$\mathbf{g} = \alpha \mathbf{T}_{d+1} + \mathbf{h}, \quad \mathbf{h} \text{ linear form in } \mathbf{T}_1, \ldots, \mathbf{T}_d.$$

Upon substituting $\alpha \mathbf{T}_{d+1}$ by $\mathbf{g} - \mathbf{h}$, we get an equivalent form

$$\mathbf{q} = \mathbf{g}_{p-1} \mathbf{T}_{d+1}^{p-1} + \mathbf{g}_{p-2} \mathbf{T}_{d+1}^{p-2} + \cdots + \mathbf{g}_0,$$

where the coefficients of the $\mathbf{g}_i$ all lie in $\mathfrak{m}^r$. Further reduction of the individual terms of $\mathbf{q}$ will eventually lead to a form only in the $\mathbf{T}_1, \ldots, \mathbf{T}_d$.

The last assertion just reflects the nature of the proof. $\square$

Remark 2.7 An a priori bound for the smallest secondary elemination degree arises as follows. Let $R = k[x_1, \ldots, x_d]$, and $I = (J, a)$ as above. Since J is generated by d forms $f_1, \ldots, f_d$ of degree n, the socle of R/J is determined by the Jacobian of the f_i which has degree $d(n-1)$. Now from the exact sequence

$$0 \to (J : a)/J \longrightarrow R/J \longrightarrow R/(J : a) \to 0,$$

the socle degree of $R/(J:a)$ is smaller than $d(n-1)$, and therefore this value gives the bound. In fact, all the examples we examined had $\epsilon+1=\min\{\, i \mid L=(L_1):\mathfrak{m}^i\}$, where ϵ is the socle degree of $R/(J:a)$.

Example 2.8 Let $R=k[x_1,x_2,x_3,x_4]$ and $\mathfrak{m}=(x_1,x_2,x_3,x_4)$.
Let $I=(x_1^3,\ x_2^3,\ x_3^3,\ x_4^3,\ x_1^2x_2+x_3^2x_4)$, $J=(x_1^3,\ x_2^3,\ x_3^3,\ x_4^3)$, and $a=x_1^2x_2+x_3^2x_4$. Then

- Hilbert series of I : $\quad 1+4t+10t^2+15t^3+15t^4+7t^5+t^6$.
- Hilbert series of $(J:a)$: $\quad 1+4t+9t^2+9t^3+4t^4+t^5$.
- Hilbert series of $I_1(\varphi)$: $\quad 1+4t+7t^2$.

A run with *Macaulay2* showed:

1. $L=(L_1):\mathfrak{m}^6$, exactly as predicted from the Hilbert function of $J:a$;
2. The calculation yielded $\operatorname{edeg}(I)=9$; in particular, Ψ_I is not birational.

The syzygetic lemmas

The following material complements and refines some known facts (see [10], [14], [15, Chapter 2]). Its main purpose is an application to almost complete intersections. Since its contents deal with arbitrary ideals, we will momentarily change notation. Let $I\subset R$ be an ideal generated by n elements $a_1,\ldots,a_n$. Consider a free presentation of I

$$R^m \xrightarrow{\varphi} R^n \longrightarrow I \to 0 \tag{6}$$

and let $(L_1)\subset L=\bigoplus_{d\geq 0} L_d\subset S=R[\mathbf{T}]=R[\mathbf{T}_1,\ldots,\mathbf{T}_n]$ denote as before the presentation ideals of the symmetric algebra and of the Rees algebra of I, respectively, corresponding to the chosen presentation.

A starting point is the following observation. Suppose $\mathbf{f}(\mathbf{T})=\mathbf{f}(\mathbf{T}_1,\ldots,\mathbf{T}_n)\in L_d$; write it as

$$\mathbf{f}(\mathbf{T}_1,\ldots,\mathbf{T}_n)=\mathbf{f}_1(\mathbf{T})\mathbf{T}_1+\cdots+\mathbf{f}_n(\mathbf{T})\mathbf{T}_n,$$

where $\mathbf{f}_i(\mathbf{T})$ is a form of degree $d-1$.

Evaluating $\mathbf{T}$ at the vector $\mathbf{a}=(a_1,\ldots,a_n)$ gives a syzygy of $\mathbf{a}$

$$z=(\mathbf{f}_1(\mathbf{a}),\ldots,\mathbf{f}_n(\mathbf{a}))\in Z_1,$$

the module of syzygies of I,

$$z\in Z_1\cap I^{d-1}R^n,$$

that is, z is a syzygy with coefficients in I^{d-1}. Note that

$$\widehat{\mathbf{f}}(\mathbf{T})=a_1\mathbf{f}_1(\mathbf{T})+\cdots+a_n\mathbf{f}_n(\mathbf{T})\in L_{d-1}\cap I\cdot S_{d-1}.$$

Conversely, any form $\mathbf{h}(\mathbf{T})$ in $L_{d-1}\cap I\cdot S_{d-1}$ can be lifted to a form $\mathbf{H}(\mathbf{T})$ in L_d with $\widehat{\mathbf{H}}(\mathbf{T}) = \mathbf{h}(\mathbf{T})$.

Such maps are referred to as *downgrading* and *upgrading*, although they are not always well-defined. However, in some case it opens the opportunity to calculate some of the higher L_d.

Here is a useful observation.

Lemma 2.9 *Let $\mathbf{f}(\mathbf{T}) \in L_d$ and write*

$$\mathbf{f}(\mathbf{T}) = \mathbf{f}_1(\mathbf{T})\mathbf{T}_1 + \cdots + \mathbf{f}_n(\mathbf{T})\mathbf{T}_n.$$

If

$$\mathbf{f}_1(\mathbf{T})a_1 + \cdots + \mathbf{f}_n(\mathbf{T})a_n = 0,$$

then $\mathbf{f}(\mathbf{T}) \in S_{d-1}L_1$.

Proof. The assumption is that $\mathbf{v} = (\mathbf{f}_1(\mathbf{T}), \ldots, \mathbf{f}_n(\mathbf{T}))$ is a syzygy of $a_1, \ldots, a_n$ over the ring $R[\mathbf{T}]$. By flatness,

$$\mathbf{v} = \sum_j \mathbf{h}_j(\mathbf{T})\mathbf{z}_j,$$

where $\mathbf{h}_j(\mathbf{T})$ are forms of degree $d-1$ and $\mathbf{z}_j \in Z_1$. Setting $\mathbf{z}_j = (z_{1j}, \ldots, z_{nj})$,

$$\mathbf{f}_i(\mathbf{T}) = \sum_j \mathbf{h}_j(\mathbf{T})z_{ij},$$

and therefore

$$\begin{aligned}
\mathbf{f}(\mathbf{T}) &= \mathbf{f}_1(\mathbf{T})\mathbf{T}_1 + \cdots + \mathbf{f}_n(\mathbf{T})\mathbf{T}_n \\
&= \sum_i (\sum_j \mathbf{h}_j(\mathbf{T})z_{ij})\mathbf{T}_i \\
&= \sum_j \mathbf{h}_j(\mathbf{T})(\sum_i z_{ij}\mathbf{T}_i) \in S_{d-1}L_1.
\end{aligned}$$

$\square$

Corollary 2.10 *Let $\mathbf{h}_j(\mathbf{T})$, $1 \leq j \leq m$, be a set of generators of $L_{d-1}\cap IS_{d-1}$. For each j, choose a form $\mathbf{F}_j(\mathbf{T}) \in L_d$ such that under one of the operations above $\widehat{\mathbf{F}}(\mathbf{T}) = \mathbf{h}_j(\mathbf{T})$. Then*

$$L_d = (\mathbf{F}_1(\mathbf{T}), \ldots, \mathbf{F}_m(\mathbf{T}), L_1S_{d-1}).$$

Proof. For $\mathbf{f}(\mathbf{T}) \in L_d$, $\mathbf{f}(\mathbf{T}) = \sum_i \mathbf{T}_i\mathbf{f}_i(\mathbf{T})$, write

$$\sum_i a_i\mathbf{f}_i(\mathbf{T}) = \sum_j c_j\mathbf{h}_j(\mathbf{T}).$$

Applying Lemma 2.9 to the polynomial

$$\mathbf{f}(\mathbf{T}) - \sum_j c_j \mathbf{F}_j(\mathbf{T})$$

will give the desired assertion. $\square$

This leads to the iterative procedure to find the equations $L = (L_1, L_2, \ldots)$ of the ideal I.

Let $Z_1 = \ker(\varphi) \subset R^n$, where φ is as in (6) and let B_1 denote the submodule of Z_1 whose elements come from the Koszul relations of the given set of generators of I. The R-module

$$\delta(I) = Z_1 \cap IR^n / B_1$$

has been introduced in [13] in order to understand the Koszul homology with coefficients in I. It is independent of the free presentation of I and as such it has been dubbed the *syzygetic module* of I. The following basic result has been proved in [14].

Lemma 2.11 ([14, 1.2]) (The syzygetic lemma) *Let I be an ideal with presentation as above. Then*

$$\delta(I) \stackrel{\phi}{\simeq} L_2/L_1S_1. \tag{7}$$

The mapping ϕ is given as follows: For $z = \sum \alpha_i \mathbf{T}_i$, $(\alpha_1, \ldots, \alpha_n) \in Z_1 \cap IR^n$, $\alpha_i = \sum_{j=1}^n c_{ij} a_j$,

$$\phi([z]) = \sum_{i,j} c_{ij} \mathbf{T}_i \mathbf{T}_j \in L_2/S_1L_1.$$

In particular, L_2/S_1L_1 is also independent of the free presentation of I and

$$\nu(L_2/S_1L_1) = \nu(\delta(I)).$$

We refer to the process of writing the α_i as linear combination of the a_j as the *extraction* of I. The knowledge of the degrees of the c_{ij} is controlled by the degrees of $\delta(I)$. Note that, quite generally, the kernel of the natural surjection $\mathrm{Sym}(I) \longrightarrow \mathcal{R}(I)$ in degree d is L_d/L_1S_{d-1}. However, a more iterative form of bookkeeping of L is through the modules L_d/S_1L_{d-1} that represent the *fresh* generators in degree d. Unfortunately, except for the case $d = 2$, one knows no explicit expressions for these modules, hence the urge for different methods to approach the problem.

Almost complete intersections

We now go back to the particular setup of almost complete intersections. As before, $(R, \mathfrak{m})$ denotes the standard graded polynomial ring $k[x_1, \ldots, x_d]$ and its irrelevant ideal and $I = (a_1, \ldots, a_d, a_{d+1})$ is an $\mathfrak{m}$-primary ideal minimally generated by $d+1$ forms of the same degree. We assume that $J = (a_1, \ldots, a_d)$ is a minimal reduction of I; set $a = a_{d+1}$.

Considerable numerical information about L_2 is readily available in this setup.

Proposition 2.12 *Let I be as above and let φ be a minimal presentation map as in* (6). *Then*

$$\nu(L_2/S_1L_1) = \nu(I_1(\varphi) : \mathfrak{m})/I_1(\varphi)).$$

In particular, if $I_1(\varphi) = \mathfrak{m}^s$, $s \geq 1$, then

$$\nu(L_2/S_1L_1) = \binom{d+s-2}{d-1}.$$

Moreover, $\delta(I)$ is generated by the last s graded components of the first Koszul homology module $H_1(I)$.

Proof. Consider the so-called syzygetic sequence of I

$$0 \to \delta(I) \longrightarrow H_1(I) \longrightarrow (R/I)^{d+1} \longrightarrow I/I^2 \to 0.$$

Note that $H_1(I)$ is isomorphic to the canonical module of R/I. Dualizing with respect to $H_1(I)$ gives the exact complex

$$H_1(I)^{d+1} \longrightarrow \mathrm{Hom}_{R/I}(H_1(I), H_1(I)) \simeq R/I \longrightarrow \mathrm{Hom}_{R/I}(\delta(I), H_1(I)) \to 0.$$

The image in R/I is the ideal generated by $I_1(\varphi)$, and since $I \subset I_1(\varphi)$, one has

$$\delta(I) \simeq \mathrm{Hom}_{R/I}(R/I_1(\varphi), H_1(I)). \tag{8}$$

It follows that $\delta(I)$ is isomorphic to the canonical module of $R/I_1(\varphi)$, and therefore $\nu(\delta(I))$ is the Cohen-Macaulay type of $R/I_1(\varphi)$.

In case $I_1(\varphi) = \mathfrak{m}^s$, $\mathrm{Hom}_{R/I}(R/\mathfrak{m}^s, H_1(I))$ cannot have a nonzero element u in higher degree as otherwise $\mathfrak{m}^{s-1}u$ would lie in the socle of $H_1(I)$, a contradiction. It follows that $\delta(I)$ is generated by $\binom{d+s-2}{d-1}$ elements. □

To help identify the generators of $\delta(I)$ requires information about the Hilbert function of $H_1(I)$. For reference we use the socle degree of $R/I_1(\varphi)$, which we denote by p. We recall that if $(1, d, a_2, \ldots, a_r)$ is the Hilbert function of R/I, that of $H_1(I)$ is $(a_r, \ldots, d, 1)$, together with an appropriate shift. Since $\delta(I)$ is a graded submodule of $H_1(I)$, it is convenient to organize a table as follows:

$$\begin{array}{rcl} H_1(I) & : & (a_r, \ldots, a_s, \ldots, a_2, d, 1) \\ \delta(I) & : & (b_p, \ldots, b_1, 1), \end{array}$$

where $b_i \leq a_i$. Note that the degrees are increasing. For example, the degree of the rth component of $H_1(I)$ is the initial degree t of the ideal $J : a$, while the degree of the pth component of $\delta(I)$ is $t + r - p$.

Balanced ideals

Let us introduce the following concept for easy reference:

Definition 2.13 An $\mathfrak{m}$-primary ideal $I \subset R$ minimally generated by $d+1$ elements of the same degree is *s-balanced* if $I_1(\varphi) = \mathfrak{m}^s$, where φ is the matrix of syzygies of I.

Note that, due to the Koszul relations, s is at most the common degrees of the generators of I. The basic result about these ideals goes as follows.

Theorem 2.14 *Let $R = k[x_1, \ldots, x_d]$ and I an almost complete intersection of finite colength generated by forms of degree n. If $I_1(\varphi) \subset \mathfrak{m}^s$, for s as large as possible, then:*

(i) *The socle degree of $H_1(I)$ is $d(n-1)$;*

(ii) $\mathfrak{m}^{d(n-1)-s+1} = I\mathfrak{m}^{(d-1)(n-1)-s}$;

(iii) *Suppose I is s-balanced. Let $r(I)$ be the degree the coefficients of L_2. Then*

$$r(I) = (d-1)(n-1) - s, \text{ and } R_{n+r(I)} = I_{n+r(I)}.$$

Proof. (i) Let J be the minimal reduction of I defined earlier. The socle degree of $H_1(I)$ is determined from the natural embedding

$$H_1(I) \simeq J : a/J \hookrightarrow R/J,$$

where the socle of R/J, which is also the socle of any of its nonzero submodules, is defined by the Jacobian determinant of d forms of degree n.

(ii) We write $H_1(I)$ and $\delta(I)$ as graded modules (set $\epsilon = d(n-1)$)

$$\begin{array}{rcl} H_1(I) & = & h_s \oplus h_{s+1} \oplus \cdots \oplus h_\epsilon \\ \delta(I) & = & f_{\epsilon-s+1} \oplus \cdots \oplus f_\epsilon, \end{array}$$

dictated by the fact that the two modules share the same socle, $h_\epsilon = f_\epsilon$. One has $\mathfrak{m}^{\epsilon-s+1}H_1(I) = 0$, hence $\mathfrak{m}^{\epsilon-s+1}R/I = 0$, or equivalently,

$$\mathfrak{m}^{\epsilon-s+1} = I\mathfrak{m}^{\epsilon-n-s+1}.$$

(iii) The degree $r(I)$ of the coefficients of L_2 is obtained from the elements of $f_{\epsilon-s+1}$, and writing them as syzygies with coefficients in I, that is

$$r(I) = \epsilon - s + 1 - n = (d-1)(n-1) - s.$$

The last assertion follows from (ii). $\square$

Let us give some consequences of this analysis which will be used later.

Corollary 2.15 *Let $R = k[x_1, \ldots, x_d]$ be a ring of polynomials, and I an almost complete intersection of finite colength generated in degree n. Suppose that I is s–balanced.*

(i) *If $d = 3$ and $s = n - 1$, then L_2 is generated by $\binom{s+1}{2}$ forms with coefficients of degree s and there are precisely n linear syzygies of degree $n - 1$, and $n \leq 7$;*

(ii) *If $d = 4$ and $s = n = 2$, then there are precisely 15 linear syzygies of degree 2.*

Proof. We begin by observing the values of $r(I)$. In case (i), $r(I) = (3-1)(n-1) - s = (3-2)(n-1) = s$, while in (ii) $r(I) = (4-1)(2-1) - 2 = 1$.

The first assertion of (i) comes from Proposition 2.12 and the value $r(I) = s$. As for the number of syzygies, according to Theorem 2.14(iv), $R_{n+s} = I_{n+s}$ in case (i), and $R_{n+1} = I_{n+1}$ in case (ii), which will permit the determination of the dimension of the linear syzygies of degree s, or higher in case (i), and for all degrees in case (ii).

Let us focus on the case $r(I) = s$. Consider the exact sequence corresponding to the generators of I,

$$R^{d+1} \xrightarrow{\pi} R \longrightarrow R/I \to 0,$$

and write ψ_s for the vector space map induced by π on the homogeneous component of degree s of R^{d+1}. We have an exact sequence of k-vector spaces and k-linear maps

$$R_s^{d+1} \xrightarrow{\psi_s} R_{n+s} \longrightarrow R_{n+s}/I_{n+s} \to 0$$

and $\ker(\psi_s)$ is the k-span of the syzygies of I of R-degree s.

One easily has

$$\dim_k(\ker(\psi_s)) = (d+1)\binom{s+d-1}{d-1} - \dim_k(I_{n+s}). \tag{9}$$

In this case, one gets

$$\dim_k(\ker(\psi_s)) = (d+1)\binom{s+d-1}{d-1} - \binom{s+n+d-1}{d-1}. \tag{10}$$

If I is moreover s-balanced for some $s \geq 1$ then it must be the case that

$$(d+1)\dim_k(\ker(\psi_s)) \geq \nu(\mathfrak{m}^s) = \dim_k(R_s) = \binom{s+d-1}{d-1}. \tag{11}$$

Suppose that $d = 3$ and I is s-balanced with $s = n - 1$. Then the equality (10) gives $\dim_k(\ker(\psi_{n-1})) = n$ while the inequality (11) easily yields $n \leq 7$.

Finally, the assertion (ii) follows immediately from the equality (10). $\square$

The numerical data alone give a bird eye vision of the generators of the graded pieces L_1 and L_2 of the ideal of equations of L. This corollary is suitable in other cases, even when $I_1(\varphi)$ is a less well packaged ideal.

3 Binary Ideals

In this section we take $d = 2$ and write $R = k[x, y]$ (instead of the general notation $R = k[x_1, x_2]$). Let $I \subset R = k[x, y]$ be an (x, y)-primary ideal generated by three forms of degree n. Suppose that I has a minimal free resolution

$$0 \to R^2 \xrightarrow{\varphi} R^3 \longrightarrow I \to 0.$$

We will assume throughout the section that the first column of φ has degree r, the other degree $s \geq r$. We note $n = r + s$.

Here we give a general format of the elimination equation of I up to a power, thus answering several questions raised in [9].

Elimination equation and degree

Set $S = R[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3]$ as before. Notice that S is standard bigraded over k. We denote by f and g the defining forms of the symmetric algebra of I, i.e., the generators of the ideal $(L_1) \subset S$ in the earlier notation. We write this in the form

$$[f, g] = [\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3] \cdot \varphi.$$

In the standard bigrading, by assumption, f has bidegree $(r, 1)$, g bidegree $(s, 1)$. According to Lemma 2.11, the component L_2 could be determined from $(L_1) : I_1(\varphi)$. In dimension two it is more convenient to get hold of a smaller quotient, $N = (L_1) : (x, y)^r$. We apply basic linkage theory to develop some properties of this ideal.

- N, being a direct link of the Cohen-Macaulay ideal $(x, y)^r$, is a perfect Cohen-Macaulay ideal of codimension two. The canonical module of S/N is generated by $(x, y)^r S/(f, g)$, so that its Cohen-Macaulay type is $r + 1$, according to Theorem 2.1.

- Therefore, by the Hilbert-Burch theorem, N is the ideal of maximal minors of an $(r+2)\times(r+1)$ matrix ζ of homogeneous forms.

- Thus, $N = (f, g) : (x, y)^r$ has a presentation $0 \to S^{r+1} \xrightarrow{\zeta} S^{r+2} \longrightarrow N \to 0$, where ζ can be written in the form

$$\zeta = \begin{bmatrix} \sigma \\ -------- \\ \tau \end{bmatrix},$$

  with σ is a $2 \times (r + 1)$ submatrix with rows whose entries are biforms of bidegree $(s - r, 1)$ and $(0, 1)$; and τ is an $r \times (r + 1)$ submatrix whose entries are biforms of bidegree $(1, 0)$.

- Since $N \subset L_2$, this shows that in L_2 there are r forms $\mathbf{h}_i$ of degree 2 in the $\mathbf{T}_i$ whose R-coefficients are forms in $(x, y)^{s-1}$.

- If $s = r$, write

$$[\mathbf{h}] = [\mathbf{h}_1 \quad \cdots \quad \mathbf{h}_r] = [x^{r-1} \quad x^{r-2}y \quad \cdots \quad xy^{r-2} \quad y^{r-1}] \cdot \mathbf{B}, \tag{12}$$

where $\mathbf{B}$ is an $r \times r$ matrix whose entries belong to $k[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3]$.

- If $s > r$, collect the $s - r$ forms $\mathbf{f} = \{x^{s-r-1}f, x^{s-r-2}yf, \ldots, y^{s-r-1}f\}$ and write

$$[\mathbf{f}; \mathbf{h}] = [\mathbf{f}; \mathbf{h}_1 \quad \cdots \quad \mathbf{h}_r] = [x^{s-1} \quad x^{s-2}y \quad \cdots \quad xy^{s-2} \quad y^{s-1}] \cdot \mathbf{B}, \tag{13}$$

where $\mathbf{B}$ is an $s \times s$ matrix whose entries belong to $k[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3]$.

A first consequence of this analysis is one of our main results:

Theorem 3.1 *In both cases,* $\det \mathbf{B}$ *is a nonzero polynomial of degree* n*.*

Proof. Case $s = r$: Suppose that $\det \mathbf{B} = 0$. Then there exists a nonzero vector $\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_r \end{bmatrix}$ whose entries are in $k[T_1, T_2, T_3]$ such that $\mathbf{B} \cdot \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_r \end{bmatrix} = 0$. Hence $[\mathbf{h}_1 \quad \cdots \quad \mathbf{h}_r] \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_r \end{bmatrix} = 0$. Since the relations of $\mathbf{h}_1, \ldots, \mathbf{h}_r$ are S-linear combinations of the columns of ζ, we get a contradiction.

The assertion on the degree follows since the degree of $\det \mathbf{B}$ is $2r = r + s = n$.

Case $s > r$: Suppose that $\det \mathbf{B} = 0$. Then there exists a nonzero vector $\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_s \end{bmatrix}$ whose entries are in $k[T_1, T_2, T_3]$ such that $\mathbf{B} \cdot \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_s \end{bmatrix} = 0$. Hence $[\mathbf{f}; \quad \mathbf{h}_1 \quad \cdots \quad \mathbf{h}_r] \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_s \end{bmatrix} = 0$. We write this relation as follows

$$\sum_{i=1}^{s-r} \mathbf{a}_\mathbf{i} x^{s-r-i} y^{i-1} f + \sum_{j=1}^{r} \mathbf{a}_{s-r+j} \mathbf{h}_j = \mathbf{a} f + \sum_{j=1}^{r} \mathbf{a}_{s-r+j} \mathbf{h}_j = 0,$$

where

$$\mathbf{a} = \sum_{i=1}^{s-r} \mathbf{a}_\mathbf{i} x^{s-r-i} y^{i-1}.$$

Since the relations of $f, \mathbf{h}_1, \ldots, \mathbf{h}_r$ are S-linear combinations of the columns of ζ,

- $\mathbf{a} \in (x, y)^{s-r} S$ and

- $\mathbf{a}_{s-r+j} \in (x,y)S$, for $1 \leq j \leq r$.

and therefore $\mathbf{a}_i \in (x,y)S$, for all i. This gives a contradiction.

The assertion on the degree follows since the degree of $\det \mathbf{B}$ is $(s-r)+2r = r+s = n$. $\square$

Example 3.2 Let $R = k[x,y]$ and I the ideal defined by $\varphi = \begin{bmatrix} x^2 & y^4 \\ xy & x^3y + x^4 \\ y^2 & xy^3 \end{bmatrix}$.

- $N = (f,g) : \mathfrak{m}^2 = (f, g, \mathbf{h}_1, \mathbf{h}_2)$, where

$$\begin{array}{rcl} f & = & x^2\mathbf{T}_1 + xy\mathbf{T}_2 + y^2\mathbf{T}_3 \\ g & = & y^4\mathbf{T}_1 + (x^3y + x^4)\mathbf{T}_2 + xy^3\mathbf{T}_3 \\ \mathbf{h}_1 & = & y^3\mathbf{T}_1^2 - x^3\mathbf{T}_2^2 - x^2y\mathbf{T}_2^2 + xy^2\mathbf{T}_1\mathbf{T}_3 - x^2y\mathbf{T}_2\mathbf{T}_3 - xy^2\mathbf{T}_2\mathbf{T}_3 \\ \mathbf{h}_2 & = & xy^2\mathbf{T}_1^2 + y^3\mathbf{T}_1\mathbf{T}_2 - x^3\mathbf{T}_2\mathbf{T}_3 - x^2y\mathbf{T}_2\mathbf{T}_3 - y^3\mathbf{T}_3^2 \end{array}$$

- $[xf \;\; yf \;\; h_1 \;\; h_2] = \mathfrak{m}^3\mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} \mathbf{T}_1 & 0 & -\mathbf{T}_2^2 & -\mathbf{T}_2\mathbf{T}_3 \\ \mathbf{T}_2 & \mathbf{T}_1 & -\mathbf{T}_2^2 - \mathbf{T}_2\mathbf{T}_3 & -\mathbf{T}_2\mathbf{T}_3 \\ \mathbf{T}_3 & \mathbf{T}_2 & \mathbf{T}_1\mathbf{T}_3 - \mathbf{T}_2\mathbf{T}_3 & \mathbf{T}_1^2 \\ 0 & \mathbf{T}_3 & \mathbf{T}_1^2 & \mathbf{T}_1\mathbf{T}_2 - \mathbf{T}_3^2 \end{bmatrix}$$

- $\det \mathbf{B}$ is the elimination equation.

Remark 3.3 The polynomials $\mathbf{h}_1, \ldots, \mathbf{h}_r$ were also obtained in [9] by a direct process involving Sylvester elimination, in the cases $s = r$ or $s = r+1$. In [9] though they did not arrive with the elements of structure–that is with their relations–provided in the Hilbert-Burch matrix. It is this fact that opens the way in the binary case to a greater generality to the ideals treated and a more detailed understanding of the ideal L.

The next result provides a secondary elimination degree for these ideals.

Corollary 3.4 *$L = (L_1) : (x,y)^{n-1}$.*

Proof. With the previous notation, let $N = L \cap Q$ be the primary decomposition of N, where Q is $(x,y)S$-primary. Writing $\beta = \det \mathbf{B}$, we then have $N : \beta = Q$. On the other hand, $(L_1, \mathbf{h}) \subset N$ by construction and $(x,y)^{s-1}S \subset (L_1, \mathbf{h}) : \beta$ since by (12 and 13) the biforms $\mathbf{h}$, or $\mathbf{f}, \mathbf{h}$, must effectively involve all monomials of degree $s-1$ in x, y. It follows that $(x,y)^{s-1}S \subset Q$, hence

$$L(x,y)^{s-1} \subset LQ \subset L \cap Q = N = (L_1) : (x,y)^r,$$

thus implying that $L(x,y)^{s+r-1} \subset (L_1)$. This shows the assertion. $\square$

Elimination equation up to a power

Theorem 3.5 *Let I be as above and $\beta = \det \mathbf{B}$. Then β is a power of the elimination equation of I.*

Proof. Let $\mathbf{p}$ denote the elimination equation of I. Since $\mathbf{p}$ is irreducible it suffices to show that β divides a power of $\mathbf{p}$.

The associated primes of $N = (L_1) : (x,y)^r$ are the defining ideal L of the Rees algebra and $\mathfrak{m}S = (x,y)S$. We have a primary decomposition

$$N = L \cap Q,$$

where Q is $\mathfrak{m}S$-primary. From the proof of Theorem 3.1, localizing at $\mathfrak{m}S$ gives $(x,y)^{s-1}S \subset Q$. (Equality will hold when $r = s$.)

The equality $N = L \cap Q = (L_1, \mathbf{h})$ implies that $(x,y)^{s-1}\mathbf{p} \subset (L_1, \mathbf{h})$. Since f, g are of bidegrees $(r,1)$ and $(s,1)$, it must be the case that each polynomial $x^i y^{s-1-i}\mathbf{p}$ lies in the span of the $(f(x,y)^{s-r-1}, \mathbf{h})$ alone. This gives a representation

$$\mathbf{p}[(x,y)^{s-1}] = [\mathbf{f}; \mathbf{h}] \cdot \mathbf{A},$$

(or simply $\mathbf{p}[(x,y)^{r-1}] = [\mathbf{h}] \cdot \mathbf{A}$, if $s = r$) where $\mathbf{A}$ is an $s \times s$ matrix with entries in $k[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3]$. Replacing $[\mathbf{f}; \mathbf{h}]$ by $[(x,y)^{s-1}] \cdot \mathbf{B}$, gives the matrix equation

$$[(x,y)^{s-1}]\big(\mathbf{B} \cdot \mathbf{A} - \mathbf{p}\mathbf{I}\big) = 0,$$

where $\mathbf{I}$ is the $s \times s$ identity matrix.

Since the minimal syzygies of $(x,y)^{s-1}$ have coefficients in (x,y), we must have

$$\mathbf{B} \cdot \mathbf{A} = \mathbf{p}\mathbf{I},$$

so that $\det \mathbf{B} \cdot \det \mathbf{A} = \mathbf{p}^r$, as desired. $\square$

4 Ternary Ideals

We outline a conjectural scenario that we expect many such ideals to conform to. Suppose I is an ideal of $R = k[x_1, x_2, x_3]$ generated by forms a_1, a_2, a_3, a of degree $n \geq 2$, with $J = (a_1, a_2, a_3)$ being a minimal reduction. This approach is required because the linkage theory method lacks the predicability of the binary ideal case.

Balanced ternary ideals

Suppose that I is $(n-1)$–balanced, where n is the degree of the generators of I. By Corollary 2.15(1), there are n linear forms $\mathbf{f}_i$, $1 \leq i \leq n$,

$$\mathbf{f}_i = \sum_{j=1}^{4} c_{ij}\mathbf{T}_j \in L_1,$$

arising from the syzygies of I of degree $n-1$. These syzygies, according to Theorem 2.1, come from the syzygies of $J:a$, which by the structure theorem of codimension three Gorenstein ideals, is given by the Pfaffians of a skew-symmetric matrix Φ, of size at most $2n-1$.

According to Proposition 2.12, there are $\binom{n}{2}$ quadratic forms $\mathbf{h}_k$ ($1 \leq k \leq \binom{n}{2}$):

$$\mathbf{h}_k = \sum_{1 \leq i \leq j \leq 4} c_{ijk} \mathbf{T}_i \mathbf{T}_j \in L_2,$$

with R-coefficients of degree $n-1$.

Picking a basis for $\mathfrak{m}^{n-1}$ (simply denoted by $\mathfrak{m}^{n-1}$), and writing the $\mathbf{f}_i$ and $\mathbf{h}_k$ in matrix format, we have

$$[\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{h}_1, \ldots, \mathbf{h}_{\binom{n}{2}}] \quad = \quad \mathfrak{m}^{n-1} \cdot \mathbf{B}, \tag{14}$$

where $\mathbf{B}$ is the corresponding content matrix (see [9]). Observe that $\det \mathbf{B}$ is either zero, or a polynomial of degree

$$n + 2\binom{n}{2} = n^2.$$

It is therefore a likely candidate for the *elimination equation*. Verification consists in checking that $\det \mathbf{B}$ is irreducible for an ideal in any given *generic* class. We will make this more precise on a quick analysis of the lower degree cases.

Theorem 4.1 *If $I \subset R = k[x_1, x_2, x_3]$ is a $(n-1)$-balanced almost complete intersection ideal generated by forms of degree n $(n \leq 7)$, then*

$$\det \mathbf{B} \neq 0.$$

Proof. Write each of the quadrics $\mathbf{h}_j$ above in the form

$$\mathbf{h}_j = c_j \mathbf{T}_4^2 + \mathbf{T}_4 \mathbf{f}(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) + \mathbf{g}(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3),$$

where c_j is a form of R of degree $n-1$, and similarly,

$$\mathbf{f}_i = c_{i4} \mathbf{T}_4 + \sum_{k=1}^{3} c_{ik} \mathbf{T}_k \in (L_1).$$

Write $\mathfrak{c} = (c_j, c_{i4})$ for the ideal of R generated by the leading coefficients of $\mathbf{T}_4$ in the $\mathbf{f}_i$'s and of $\mathbf{T}_4^2$ in the $\mathbf{h}_j$'s. It is apparent that if $\mathfrak{c} = \mathfrak{m}^{n-1}$, there will be a non-cancelling term $\mathbf{T}_4^{n^2}$ in the expansion of $\det \mathbf{B}$.

To argue that indeed $\mathfrak{c} = \mathfrak{m}^{n-1}$ is the case, assume otherwise. Since the $\mathbf{f}_i$ are minimal generators that contribute to $(J:a)$, we may assume that the c_{i4} are linearly independent. This implies that we may replace one of the $\mathbf{h}_j$ by a form

$$\begin{aligned} \mathbf{h} &= \mathbf{T}_4 \mathbf{f}(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) + \mathbf{g}(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) \\ &= \mathbf{T}_4(r_1 \mathbf{T}_1 + r_2 \mathbf{T}_2 + r_3 \mathbf{T}_3) + \mathbf{T}_1 \mathbf{g}_1 + \mathbf{T}_2 \mathbf{g}_2 + \mathbf{T}_3 \mathbf{g}_3, \end{aligned}$$

where the r_i are $(n-1)$-forms in R and the $\mathbf{g}$'s are $\mathbf{T}$-linear involving only $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$.

Evaluate now $\mathbf{T}_i$ at the corresponding generator of I to get

$$(ar_1 + \mathbf{g}_1(a_1, a_2, a_3))\, a_1 + (ar_2 + \mathbf{g}_2(a_1, a_2, a_3))\, a_3 + (ar_3 + \mathbf{g}_3(a_1, a_2, a_3))\, a_3 = 0,$$

a syzygy of the ideal $J = (a_1, a_2, a_3)$. Since J is a complete intersection, $ar_i + \mathbf{g}_i(a_1, a_2, a_3) = \mathbf{u}_i(a_1, a_2, a_3) \in J$ for $i = 1, 2, 3$, with $\mathbf{u}_i$ a linear form in $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ with coefficients in R. These are syzygies of the generators of I, so lifting back to 1-forms in $\mathbf{T}$ and substituting yields $\mathbf{h} = \mathbf{h}' + \mathbf{k}$, where

$$\begin{aligned} \mathbf{h}' &= (r_1\mathbf{T}_4 + \mathbf{g}_1(\mathbf{T}_2, \mathbf{T}_2, \mathbf{T}_3) - \mathbf{u}_1(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3))\, \mathbf{T}_1 \\ &+ (r_2\mathbf{T}_4 + \mathbf{g}_2(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) - \mathbf{u}_2(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3))\, \mathbf{T}_2 \\ &+ (r_3\mathbf{T}_4 + \mathbf{g}_3(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) - \mathbf{u}_3(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3))\, \mathbf{T}_3 \end{aligned}$$

is an element of (L_1), and because $\mathbf{h}$ is a relation then so is the term $\mathbf{k}$. But the latter only involves $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$, hence it belongs to the defining ideal of the symmetric algebra of the complete intersection J. But the latter is certainly contained in (L_1). Summing up we have found that $\mathbf{h} \in (L_1)$, which is a contradiction since $\mathbf{h}$ is a minimal generator of L_2/S_1L_1. $\square$

Example 4.2 This example shows, in particular, that there are $(n-1)$–balanced ideals $I \subset k[x_1, x_2, x_3]$ generated in degree n such that the corresponding map Ψ_I is not birational onto its image.

Let $I = (J, a)$, where $J = (x_1^3,\, x_2^3,\, x_3^3)$ and $a = x_1x_2x_3$.

- The Hilbert series of $R/(J : a)$ is $1 + 3t + 3t^2 + t^3$.
- $I_1(\varphi) = \mathfrak{m}^2$, i.e., I is 2–balanced.
- $(L_1) : \mathfrak{m}^4 = (L_1) : \mathfrak{m}^5$ while $(L_1) : \mathfrak{m}^3 \neq (L_1) : \mathfrak{m}^4$
- Equations of I:
$$\left\{ \begin{array}{l} L = (L_1, L_2, L_3) \\ \nu(L_1) = 6; \nu(L_2) = 3; L_3 = kF, \end{array} \right.$$
where $F = -\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3 + \mathbf{T}_4^3$ is the elimination equation; in particular, the corresponding map Ψ_I is not birational onto its image.
- Let $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ be generators of L_1 with coefficients in $\mathfrak{m}^2$ and $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \in L_2$ as previously described. Writing $[\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = \mathfrak{m}^2\mathbf{B}$ as in (14), one has
$$\mathbf{B} = \begin{bmatrix} 0 & 0 & -\mathbf{T}_4 & \mathbf{T}_3 & 0 & 0 \\ 0 & -\mathbf{T}_4 & 0 & 0 & \mathbf{T}_2 & 0 \\ -\mathbf{T}_4 & 0 & 0 & 0 & 0 & \mathbf{T}_1 \\ \mathbf{T}_2\mathbf{T}_3 & 0 & 0 & 0 & 0 & -\mathbf{T}_4^2 \\ 0 & \mathbf{T}_1 T_3 & 0 & 0 & -\mathbf{T}_4^2 & 0 \\ 0 & 0 & \mathbf{T}_1\mathbf{T}_2 & -\mathbf{T}_4^2 & 0 & 0 \end{bmatrix}$$
and $\det \mathbf{B} = F^3$.

Remark 4.3 To strengthen Theorem 4.1 to the assertion that $\det \mathbf{B}$ is a power of the elimination equation, one needs more understanding of the ideal $(L_1) : \mathfrak{m}^{n-1}$. Here is one such instance.

Proposition 4.4 *Let $I \subset R = k[x_1, x_2, x_3]$ be an $(n-1)$-balanced almost complete intersection ideal generated by forms of degree n. Keeping the notation introduced at the beginning of this section, we obtain the following :*

(i) *$2n-2$ is a secondary elimination degree of I.*

(ii) *If $(L_1) : \mathfrak{m}^{n-1} = (\mathbf{f}, \mathbf{h}) = (\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{h}_1, \ldots, \mathbf{h}_{\binom{n}{2}})$, then $\det \mathbf{B}$ is a power of the elimination equation.*

Proof. Let $(L_1) : \mathfrak{m}^{n-1} = L \cap Q$, where Q an $\mathfrak{m}S$-primary ideal. As in (14) one has $[\mathbf{f}, \mathbf{h}] = \mathfrak{m}^{n-1} \cdot \mathbf{B}$. Notice that $(\mathbf{f}, \mathbf{h}) \subset ((L_1) : \mathfrak{m}^{n-1})$. Write $\beta = \det \mathbf{B}$. Since $\det \mathbf{B} \neq 0$, it follows that

$$\mathfrak{m}^{n-1} S \subset (\mathbf{f}, \mathbf{h}) : \beta \subset ((L_1) : \mathfrak{m}^{n-1}) : \beta = Q.$$

This implies that

$$L \cdot \mathfrak{m}^{n-1} \subset LQ \subset (L_1) : \mathfrak{m}^{n-1}.$$

Hence $L = (L_1) : \mathfrak{m}^{2n-2}$, which proves (i).

Now suppose that $(L_1) : \mathfrak{m}^{n-1} = (\mathbf{f}, \mathbf{h})$. Let $\mathbf{p}$ be the elimination equation. By (i), we have

$$\mathbf{p} \in (L_1) : \mathfrak{m}^{2n-2} = ((L_1) : \mathfrak{m}^{n-1}) : \mathfrak{m}^{n-1}.$$

Therefore $\mathbf{p}\mathfrak{m}^{n-1} \subset (\mathbf{f}, \mathbf{h})$, which gives a representation

$$\mathbf{p}[\mathfrak{m}^{n-1}] = [\mathbf{f}, \mathbf{h}] \cdot \mathbf{A},$$

where $\mathbf{A}$ is a square matrix with entries in S. Replacing $[\mathbf{f}, \mathbf{h}]$ by $[\mathfrak{m}^{n-1}] \cdot \mathbf{B}$, gives the matrix equation

$$[\mathfrak{m}^{n-1}]\big(\mathbf{B} \cdot \mathbf{A} - \mathbf{p}\mathbf{I}\big) = 0,$$

where $\mathbf{I}$ is the identity matrix. Since the minimal syzygies of $\mathfrak{m}^{r-1}$ have coefficients in $\mathfrak{m}$, we must have

$$\mathbf{B} \cdot \mathbf{A} = \mathbf{p}\mathbf{I},$$

so that $\det \mathbf{B} \cdot \det \mathbf{A} = \mathbf{p}^m$, for some integer m; this proves (ii). $\square$

Ternary quadrics

We apply the preceding discussion to the situation where the ideal I is generated by 4 quadrics of the polynomial ring $R = k[x_1, x_2, x_3]$. The socle of R/J is generated by the Jacobian determinant of a_1, a_2, a_3, which implies that $\lambda(I/J) \geq 2$. Together we obtain that $\lambda(R/I) = 6$. The Hilbert function of R/I is $(1, 3, 2)$. Since we cannot have $u\mathfrak{m} \subset I$ for some 1-form u, the type of I is 2 and its socle is generated in degree two.

The canonical module of R/I satisfies $\lambda((J : a)/J) = 6$, hence $\lambda(\mathfrak{m}/(J : a)) = 1$, that is to say $J : a = (v_1, v_2, v_3^2)$, where the v_i are linearly independent 1-forms. Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be the linear syzygies of I induced by v_1 and v_2 respectively. R/I has a free presentation

$$0 \to R^2 \longrightarrow R^5 \xrightarrow{\varphi} R^4 \longrightarrow R/I \to 0.$$

The ideal $I_1(\varphi)$ is either $\mathfrak{m}$ or (v_1, v_2, v_3^2). In the first case, $\delta(I)$ is the socle of $H_1(I)$, an element of degree 3, therefore its image in L_2 is a 2-form $\mathbf{h}_1$ with linear coefficients. In particular reduction number cannot be two. Putting it together with the two linear syzygies $\mathbf{f}_1, \mathbf{f}_2$ of I, we

$$[\mathbf{f}_1, \mathbf{f}_2, \mathbf{h}_1] = [x_1, x_2, x_3] \cdot \mathbf{B},$$

where $\mathbf{B}$ is a 3×3 matrix with entries in $k[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4]$, of column degrees $(1, 1, 2)$. The quartic $\det \mathbf{B}$, is the elimination equation of I.

In the other case, $\delta(I)$ has degree two, so its image in L_2 is a form with coefficients in the field. The corresponding mapping Ψ_I is not birational.

We sum up the findings in this case:

Theorem 4.5 *Let $R = k[x_1, x_2, x_3]$ and let I be an $\mathfrak{m}$-primary almost complete intersection generated by quadrics. Then*

(i) *If $I_1(\varphi) = \mathfrak{m}$ the corresponding mapping Ψ_I is birational onto its image.*

(ii) *If $I_1(\varphi) \neq \mathfrak{m}$ then $I_1(\varphi) = (v_1, v_2, v_3^2)$, where v_1, v_2, v_3 are linearly independent 1-forms, and the mapping Ψ_I is not birational onto its image.*

Here is a sufficiently general example fitting the first case in the above theorem (same behavior as 4 random quadrics):

$$J = (x_1^2, x_2^2, x_3^2), \quad a = x_1x_2 + x_1x_3 + x_2x_3. \tag{15}$$

A calculation shows that $\det \mathbf{B}$ is irreducible.

An example which is degenerate (non-birational) as in the second case above is

$$J = (x_1^2, x_2^2, x_3^2), \quad a = x_1x_2. \tag{16}$$

Then $(L_1, x_3(\mathbf{T}_1\mathbf{T}_2 - \mathbf{T}_4^2)) = (L_1) : \mathfrak{m} \subsetneq (L_1) : I_1(\varphi) = L$. Let $\mathbf{h} = x_3(\mathbf{T}_1\mathbf{T}_2 - \mathbf{T}_4^2)$ and write

$$[\mathbf{f}_1, \mathbf{f}_2, \mathbf{h}] = [x_1, x_2, x_3] \cdot \mathbf{B}.$$

Then $\det \mathbf{B}$ is a square of the elimination equation $\mathbf{p} = \mathbf{T}_1\mathbf{T}_2 - \mathbf{T}_4^2$, so we still recover the elimination equation from $\mathbf{B}$.

Ternary cubics and quartics

Let I be an ideal generated by 4 cubics and suppose that I is 2-balanced (i.e., $I_1(\varphi) = \mathfrak{m}^2$). Using this (see the beginning of Section 4) and the fact that $J : a$ is a codimension 3 Gorenstein ideal, it follows that $J : a$ is minimally generated by the Pfaffians of a skew-symmetric matrix of sizes 3 or 5.

In the first case, $J : a$ is generated by 3 quadrics and I is a Northcott ideal. In the second case, $J : a$ cannot be generated by 5 quadrics, as its Hilbert function would be $(1, 3, 1)$ and therefore the Hilbert function of R/I would have to be

$$(1, 3, 6, 7, 6, 3, 1) - (0, 0, 0, 1, 3, 1) = (1, 3, 6, 6, 3, 2, 1),$$

giving that the canonical module of R/I had a generator in degree 0. Thus $J : a$ must be generated by 3 quadrics and 2 cubics.

Remark 4.6 Let $R = k[x_1, x_2, x_3]$ and I a balanced $\mathfrak{m}$-primary almost complete intersection. We expect that with an appropriate notion of genericity the following assertions will hold.

1. If I generated by cubics and $I_1(\varphi) = \mathfrak{m}^2$, the polynomial $\det \mathbf{B}$, defined by the equation (14), is the elimination equation of I.

2. If I generated by quartics and $I_1(\varphi) = \mathfrak{m}^3$, the polynomial $\det \mathbf{B}$, defined by the equation (14), is the elimination equation of I.

In what follows we give examples to cover this expected behavior: the first two are instances of (1), while the third illustrates (2).

Example 4.7 Let $I = (J, a)$, where $J = (x_1^3 + x_2^2 x_3,\, x_2^3 + x_1 x_3^2,\, x_3^3 + x_1^2 x_2)$ and $a = x_1 x_2 x_3$.

- $J : a$ is a complete intersection
- $(L_1) : \mathfrak{m}^4 = (L_1) : \mathfrak{m}^5$ while $(L_1) : \mathfrak{m}^3 \neq (L_1) : \mathfrak{m}^4$
- Equations of I:

$$\begin{cases} L = (L_1, L_2, L_5, L_9) \\ \nu(L_1) = 6;\ \nu(L_2) = 3;\ \nu(L_5) = 15;\ L_9 = k \det \mathbf{B}, \end{cases}$$

  where $\det \mathbf{B}$, obtained as in (14), is of degree 9, hence must be the elimination equation and the mapping Ψ_I is birational onto its image.

Example 4.8 Let $I = (J, a)$, where $J = (x_1^3,\, x_2^3,\, x_3^3)$ and $a = x_1^2 x_2 + x_2^2 x_3 + x_1 x_3^2$.

- $J : a$ is Gorenstein.
- $(L_1) : \mathfrak{m}^4 = (L_1) : \mathfrak{m}^5$ while $(L_1) : \mathfrak{m}^3 \neq (L_1) : \mathfrak{m}^4$.

- Equations of I:

$$\begin{cases} L = (L_1, L_2, L_4, L_9) \\ \nu(L_1) = 7;\ \nu(L_2) = 3;\ \nu(L_4) = 6;\ L_9 = k \det \mathbf{B}, \end{cases}$$

  where $\det \mathbf{B}$, obtained as in (14), is of degree 9, hence must be the elimination equation and the mapping Ψ_I is birational onto its image.

Example 4.9 Let $I = (J, a)$, where $J = (x_1^4,\ x_2^4,\ x_3^4)$ and $a = x_1^3 x_2 + x_2^3 x_3 + x_1 x_3^3$.

This follows the pattern of the previous example, with $J : a$ is Gorenstein. The structure of L is now involved, however the principle in (14) still works and gives $\det \mathbf{B}$ of degree 16, hence must be the elimination equation and the mapping Ψ_I is birational onto its image.

5 Quaternary Forms

In this Section, we set $R = k[x_1, x_2, x_3, x_4]$ with $\mathfrak{m} = (x_1, x_2, x_3, x_4)$.

Quaternary quadrics

Let I be generated by 5 quadrics a_1, a_2, a_3, a_4, a_5 of R, $J = (a_1, a_2, a_3, a_4)$, and $a = a_5$. The analysis of this case is less extensive than the case of ternary ideals. Let us assume that $I_1(\varphi) = \mathfrak{m}^2$ – that is, the balancedness exponent equals the degree of the generators, a case that occurs generically in this degree.

We claim that the Hilbert function of R/I is $(1, 4, 5)$. Since we cannot have $u\mathfrak{m} \subset I$ for some 1-form u, its socle is generated in degree two or higher. First, we argue that $J : a \neq \mathfrak{m}^2$; in fact, otherwise R/I would be of length 11 and type 6 as $\mathfrak{m}^2/J$ is its canonical module. But then the Hilbert function of R/I would be $(1, 4, 5, 1)$, and the last two graded components would be in the socle, which is impossible.

Thus it must be the case that $\lambda(R/I) = 10$ and the Hilbert function of $H_1(I)$ is $(5, 4, 1)$. Note that $\nu(\delta(I)) = 4$.

The last two graded components are of degrees 3 and 4. In degree 3 it leads to 4 forms $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ in L_2, with linear coefficients:

$$[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4] = [x_1, x_2, x_3, x_4] \cdot \mathbf{B}.$$

Theorem 5.1 $\det \mathbf{B} \neq 0$.

Proof. We follow the pattern of the proof of Theorem 4.1. Write each of the quadrics $\mathbf{q}_i$ as

$$\mathbf{q}_i = c_i \mathbf{T}_5^2 + \mathbf{T}_5 \mathbf{f}(\mathbf{T}_1 \ldots, \mathbf{T}_4) + \mathbf{g}(\mathbf{T}_1, \ldots, \mathbf{T}_4),$$

where c_i is a linear form of R. If $(c_1, c_2, c_3, c_4) = \mathfrak{m}$ we can take the c_i for indeterminates in order to obtain the corresponding $\det \mathbf{B}$. In this case the occurrence of a non-cancelling term $\mathbf{T}_5^8$ in $\det \mathbf{B}$ would be clear.

By contradiction, assume that the forms c_i are not linearly independent. In this case, we could replace one of the $\mathbf{q}_i$ by a form

$$\mathbf{q} = \mathbf{T}_5\mathbf{f}(\mathbf{T}_1, \ldots, \mathbf{T}_4) + \mathbf{g}(\mathbf{T}_1, \ldots, \mathbf{T}_4).$$

Keeping in mind that $\mathbf{q}$ is a minimal generator we are going to argue that $\mathbf{q} \in (L_1)$. For that end, write the form $\mathbf{q}$ as

$$\mathbf{q} = \mathbf{T}_5(r_1\mathbf{T}_1 + \cdots + r_4\mathbf{T}_4) + \mathbf{T}_1\mathbf{g}_1 + \cdots + \mathbf{T}_4\mathbf{g}_4,$$

where the r_i are 1-forms in R and $\mathbf{g}_i$'s are $\mathbf{T}$–linear involving only $\mathbf{T}_1, \ldots, \mathbf{T}_4$.

Evaluate now the leading $\mathbf{T}_i$ at the ideal to get the syzygy

$$\mathbf{h} = a(r_1\mathbf{T}_1 + \cdots + r_4\mathbf{T}_4) + a_1\mathbf{g}_1 + \cdots + a_4\mathbf{g}_4,$$

of I, but actually of the ideal J. Since J is a complete intersection, all the coefficients of $\mathbf{h}$ lie in J. This implies that $r_i a \in J$ for all r_i. But this is impossible since $J : a \in I_1(\varphi) = \mathfrak{m}^2$, unless all $r_i = 0$. This would imply that $\mathbf{q} \in (L_1)$, as asserted. $\square$

Theorem 5.2 *Let $R = k[x_1, x_2, x_3, x_4]$ and I an $\mathfrak{m}$-primary almost complete intersection. If I is generated by quadrics and $I_1(\varphi) = \mathfrak{m}^2$, the polynomial* $\det \mathbf{B}$*, defined by the equation* (14) *is divisible by the elimination equation of I.*

Primary decomposition

We now derive a value for the secondary elimination degree via a primary decomposition.

Proposition 5.3 *Let $R = k[x_1, x_2, x_3, x_4]$ and I an $\mathfrak{m}$-primary almost complete intersection. Suppose that I is generated by quadrics and $I_1(\varphi) = \mathfrak{m}^2$. Then*

$$(L_1) : \mathfrak{m}^2 = L \cap \mathfrak{m}S.$$

In particular,

$$L = (L_1) : \mathfrak{m}^3.$$

By Theorem 2.6, it will suffice to show:

Lemma 5.4 $\mathfrak{m}^3 = (J : a)\mathfrak{m}$.

Proof. We make use of the diagram

$$
\begin{tikzcd}
& R \arrow[d, dash] & \\
& \mathfrak{m} \arrow[d, dash, "4"] & \\
& \mathfrak{m}^2 \arrow[dl, dash, "1"'] \arrow[dd, dash, "11"] \arrow[dr, dash, "\geq 5"] & \\
J : a \arrow[dr, dash] & & I \arrow[dl, dash] \\
& J &
\end{tikzcd}
$$

Let $I = (a_1, \ldots, a_5)$, $J = (a_1, \ldots, a_4)$ a minimal reduction of I, and $a = a_5$. Since $(J : a)$ is a Gorenstein ideal, it cannot be $\mathfrak{m}^2$. On the other hand, $\lambda((J : a)/J) = \lambda(R/I) \geq 10$. Thus $\lambda(\mathfrak{m}^2/(J : a)) = 1$ and $(J : a) : \mathfrak{m} = \mathfrak{m}^2$ defines the socle of $R/(J : a)$. The Hilbert function of $R/(J : a)$ is then $(1, 4, 1)$ which implies that $\mathfrak{m}^3 = (J : a)\mathfrak{m}$. $\square$

Examples of quaternary quadrics

We give a glimpse of the various cases.

Example 5.5 Let $J = (x_1^2, x_2^2, x_3^2, x_4^2)$ and $a = x_1x_2 + x_2x_3 + x_3x_4$

- $\nu(J : a) = 9$ (all quadrics)
- $I_1(\varphi) = \mathfrak{m}^2$
- As explained above, there are four Rees equations q_j of bidegree $(1, 2)$ coming from the syzygetic principle. One has:

$$
\left\{ \begin{array}{l} L = (L_1, L_2, L_9), \\ \nu(L_1) = 15; \nu(L_2) = 4; L_9 = k\mathbf{p}, \end{array} \right.
$$

  where $\mathbf{p} = \det \mathbf{B}$ has degree 8, hence is the elimination equation and the corresponding map is birational.

The following example is similar to the above example, including the syzygetic principle, except that F is now the square root of $\det \mathbf{B}$.

Example 5.6 Let $J = (x_1^2, x_2^2, x_3^2, x_4^2)$ and $a = x_1x_2 + x_3x_4$.

- $\nu(J : a) = 9$ (all quadrics)
- $I_1(\varphi) = \mathfrak{m}^2$

- Equations of I:

$$\begin{cases} L = (L_1, L_2, L_4) \\ \nu(L_1) = 15; \nu(L_2) = 4; L_4 = k\mathbf{p}, \end{cases}$$

  where $\deg(\mathbf{p}) = 4$, hence Ψ_I is not birational onto its image.

- Letting $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ be generating forms in L_2, with linear coefficients, write

$$[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4] = [x_1, x_2, x_3, x_4] \cdot \mathbf{B}.$$

  Then $\det \mathbf{B} = \mathbf{p}^2$.

Next is an example where the normal syzygetic procedure fails, but one can apply one more step to get the elimination equation. We will accordingly give the details of the calculation.

Example 5.7 Let $I = (J, a)$, where $J = (x_1^2,\, x_2^2,\, x_3^2,\, x_4^2)$ and $a = x_1x_2 + x_2x_3 + x_3x_4 + x_1x_4$.

- $\nu(J : a) = 4$ (complete intersection of 2 linear equations and 2 quadrics)
- $I_1(\varphi) = \mathfrak{m}$ (i.e., 1-balanced, not 2-balanced)
- $\nu(Z_1) = 9$ (2 linear syzygies and 7 quadratic ones)
- Hilbert series of $R/(J : a) :\, 1 + 2t + t^2$.
- $\mathrm{edeg}(I) = 8 = 2^3$ (i.e., birational)
- The one step syzygetic principle does not work to get the elimination equation: One has:

$$\begin{cases} L = (L_1, L_2, L_3, L_8), \\ \nu(L_1) = 9; \nu(L_2) = 1; \nu(L_3) = 2, L_8 = k\mathbf{p}, \end{cases}$$

  Nevertheless there are relationships among these numbers that are understood from the syzygetic discussion. Thus Theorem 2.14(iii) implies that L_2/S_1L_1 is generated by one form with linear coefficients.

- Let f_1 and f_2 be in L_1 with linear coefficients, i.e.,

  ◊ $f_1 = (x_1 + x_3)(\mathbf{T}_2 - \mathbf{T}_4) + (x_2 - x_4)(-\mathbf{T}_5)$

  ◊ $f_2 = (x_1 - x_3)(-\mathbf{T}_5) + (x_2 + x_4)(\mathbf{T}_1 - \mathbf{T}_3)$

- Let $\mathbf{h} + L_1S_1$ be a generator of L_2/L_1S_1. We observed that, as a coset, $\mathbf{h}$ can be written as $\mathbf{h} = \mathbf{h}_1 + S_1L_1$ and $\mathbf{h} = \mathbf{h}_2 + S_1L_1$, with $\mathbf{h}_1$ and $\mathbf{h}_2$ forms of bidegree $(2, 2)$, with R-content contained in the contents f_1 and f_2, respectively.

- Let $\mathbf{h}_1 = (-2x_1\mathbf{T}_4\mathbf{T}_5 - 2x_3\mathbf{T}_4\mathbf{T}_5 + x_4\mathbf{T}_5^2)(x_1 + x_3) + (x_3\mathbf{T}_1\mathbf{T}_2 - x_3\mathbf{T}_2\mathbf{T}_3 - x_3\mathbf{T}_1\mathbf{T}_4 + x_3\mathbf{T}_3\mathbf{T}_4 - x_4\mathbf{T}_1\mathbf{T}_5 + 2x_2\mathbf{T}_3\mathbf{T}_5 - x_4\mathbf{T}_3\mathbf{T}_5 - x_3\mathbf{T}_5^2)(x_2 - x_4)$. Then $\mathbf{h}_1 \in L_2$ and $2\mathbf{h} + \mathbf{h}_1 \in L_1S_1$. Hence we may choose $\mathbf{h}_1 + L_1S_1$ to be a generator of L_2/L_1S_1.

- Let $\mathbf{h}_2 = (x_4\mathbf{T}_1\mathbf{T}_2 - x_4\mathbf{T}_2\mathbf{T}_3 - x_4\mathbf{T}_1\mathbf{T}_4 + x_4\mathbf{T}_3\mathbf{T}_4 - x_3\mathbf{T}_2\mathbf{T}_5 + 2x_1\mathbf{T}_4\mathbf{T}_5 - x_3\mathbf{T}_4\mathbf{T}_5 - x_4\mathbf{T}_5^2)(x_1 - x_3) + (-2x_2\mathbf{T}_3\mathbf{T}_5 - 2x_4\mathbf{T}_3\mathbf{T}_5 + x_3\mathbf{T}_5^2)(x_2 + x_4)$. Then $\mathbf{h}_2 \in L_2$ and $2\mathbf{h} + \mathbf{h}_2 \in L_1S_1$. Hence we may choose $\mathbf{h}_2 + L_1S_1$ to be a generator of L_2/L_1S_1.

- Write
$$[f_1 \;\; \mathbf{h}_1] = [x_1 + x_3 \;\; x_2 - x_4]\mathbf{B}_1 \quad \text{and} \quad [f_2 \;\; \mathbf{h}_2] = [x_1 - x_3 \;\; x_2 + x_4]\mathbf{B}_2$$
Then $\det \mathbf{B}_1$ and $\det \mathbf{B}_2$ form a minimal generating set of L_3/S_1L_2.

- Write $[f_1 \;\; f_2 \;\; \det \mathbf{B}_1 \;\; \det \mathbf{B}_2] = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]\mathbf{B}$.
At the outcome $\det \mathbf{B} = \mathbf{p}$ is of degree 8, hence this is again birational.

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